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On the Browder's Theorem of an Elementary Operator

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Abstract: Let H be an infinite complex Hilbert space and consider two bounded linear operators $A, B \in L(H)$. Let $L_A \in L(L(H))$ and $R_B \in L(L(H))$ be the left and the right multiplication operators, respectively, and denote by $\mathrm{d}_{A,B} \in L(L(H))$ either the elementary operator $\Delta_{A,B}(X) = (L_A R_B - I)(X) = AXB - X$ or the generalized derivation $\delta_{A,B}(X) = (L_A - R_B)(X) = AX - XB$. This paper is concerned with the problem of the transference of Browder's theorem from operators A and B to their elementary operator $\mathrm{d}_{A,B}$. We give necessary and sufficient conditions for $\mathrm{d}_{A,B}$ to satisfy Browder's theorem. Some applications for completely hereditarily normaloid operators are given.

Key words: Browder's theorem, elementary operator, hereditarily polaroid operators.

AMS *Subject Class.* (2010): 47A10, 47A53, 47B47.

1. Introduction

Let H be an infinite complex Hilbert space and consider two bounded linear operators $A, B \in L(H)$. Let $L_A \in L(L(H))$ and $R_B \in L(L(H))$ be the left and the right multiplication operators, respectively. We denote by $\mathrm{d}_{A,B} \in L(L(H))$ either the elementary operator $\Delta_{A,B}(X) = (L_A R_B - I)(X) = AXB - X$ or the generalized derivation $\delta_{A,B}(X) = (L_A - R_B)(X) = AX - XB$. The problem of the transmission of Weyl type theorems from A and B to $\mathrm{d}_{A,B}$ was studied by numerous mathematicians, see [5, 8, 19, 20, 21, 17] and the references therein.

The main objective of this paper is the transmission of Browder's theorem from A and B to $\mathrm{d}_{A,B}$. In the third section of this paper, we characterize the Browder spectrum of $\mathrm{d}_{A,B}$ by showing that

$$\begin{aligned}\sigma_b(\delta_{A,B}) &= \big(\sigma_b(A) - \sigma(B)\big) \cup \big(\sigma(A) - \sigma_b(B)\big)\,,\\ \sigma_b(\Delta_{A,B}) &= \sigma(A)\sigma_b(B) \cup \sigma_b(A)\sigma(B) - \{1\}\,.\end{aligned}$$

Moreover, when A and B satisfies Browder's theorem we give necessary and sufficient conditions for $\mathrm{d}_{A,B}$ to satisfy Browder's theorem. We also give answer to a question posed in [5]. In the last section we give an application to completely hereditarily normaloid operators and then extend some well-known results

2. Notation and terminology

Let $T \in L(X)$ be a bounded linear operator on an infinite dimensional complex Banach space X and denote by $\alpha(T)$ the dimension of the kernel $\ker T$, and by $\beta(T)$ the codimension of the range $R(T)$. $T \in L(X)$ is said to be a Fredholm operators if $R(T)$ is closed and $\alpha(T)$ and $\beta(T)$ are both finite. In this case the index of T is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. An operator $T \in L(X)$ is said to be Weyl operator if it is Fredholm operator of index zero. The essentiel (Fredholm) spectrum $\sigma_e(T)$ and the Weyl spectrum $\sigma_W(T)$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm operator}\},$$
$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl operator}\}.$$

Recall that the ascent $p(T)$ of an operator T is defined by

$$p(T) = \inf\left\{n \in \mathbb{N} : \ker T^n = \ker T^{n+1}\right\}$$

and the descent

$$q(T) = \inf\left\{n \in \mathbb{N} : R\big(T^n\big) = R\big(T^{n+1}\big)\right\},$$

with $\inf \emptyset = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. $T \in L(X)$ is said to be Browder operator if T is Fredholm operator with finite ascent and descent. Note that if T is Browder then T is Weyl. The Browder spectrum $\sigma_b(T)$ is defined by

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder operator}\}.$$

We recall that

$$\sigma_b(T) = \sigma_e(T) \cup \operatorname{acc} \sigma(T),$$

here and in the sequel we shall denote by $\operatorname{acc} D$ and $\operatorname{iso} D$, the set of accumulation points and the set of isolated points of $D \subset \mathbb{C}$, respectively.

For $T \in L(X)$ and a nonnegative integer n define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$. If for some integer n the range space $R(T^n)$ is closed and T_n is Fredholm operator, then T is called B-Fredholm operator. In this case the index of T is defined as the index of the Fredholm operator T_n, see [6]. An operator $T \in L(X)$ is said to be B-Weyl operator if it is B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not B-Weyl operator}\}.$$

We say that generalized Weyl's theorem holds for T if

$$\sigma(T) \backslash \sigma_{BW}(T) = E(T),$$

where $E(T)$ is the set of isolated eigenvalues of T.

A weaker version of generalized Weyl's theorem was given in [7], T is said to satisfy generalized Browder's theorem if

$$\sigma(T) \backslash \sigma_{BW}(T) = \Pi(T),$$

where $\Pi(T)$ is the set of poles of the resolvent of T.

M. Berkani [6, Theorem 4.5] has shown that every normal operator T acting on a Hilbert space satisfies generalized Weyl's theorem. This gives a generalization of the classical Weyl's theorem. Recall that the classical Weyl's theorem asserts that for every normal operator T acting on a Hilbert space,

$$\sigma(T) \backslash \sigma_W(T) = E_0(T),$$

where $E_0(T)$ is the set of isolated eigenvalues of finite multiplicity of T [23].

A weaker version for Weyl's theorem was introduced in [16], Browder's theorem holds for T if

$$\sigma(T) \backslash \sigma_W(T) = \Pi_0(T),$$

where $\Pi_0(T)$ is the set of all isolated points of $\sigma(T)$ for which the corresponding spectral projection has finite dimensional range, or equivalently necessary and sufficient condition for T to satisfy Browder's theorem is the identity $\sigma_W(T) = \sigma_b(T)$. In [2, Theorem 2.1] the authors proved that Browder's and generalized Browder's theorems are equivalent.

Definition 2.1. An operator $T \in L(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc $\mathbb{D}$ centered at λ_0, the only analytic function $f : \mathbb{D} \to X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every $\lambda \in \mathbb{C}$.

Evidently, every operator T, as well as its dual T^*, has SVEP at every point in $\partial\sigma(T)$, where $\partial\sigma(T)$ is the boundary of the spectrum $\sigma(T)$, in particular at every isolated point of $\sigma(T)$.

3. Browder's theorem for $\mathrm{d}_{A,B}$

Lemma 3.1. *Let K and L two compact subset of $\mathbb{C}$. Then*

$$\mathrm{acc}(K-L) = (\mathrm{acc}\, K - L) \cup (K - \mathrm{acc}\, L)\,.$$

Proof. Let $\lambda \in \mathrm{acc}\, K - L$. Then $\lambda = \mu - \nu$ with $\mu \in \mathrm{acc}\, K$ and $\nu \in L$. Hence there exist a sequence $\mu_n \in K$ which converge to μ. Now $\lambda_n = \mu_n - \nu \in (K-L)$ and converge to λ. Thus $\lambda \in \mathrm{acc}(K-L)$. Then $\mathrm{acc}\, K - L \subseteq \mathrm{acc}(K-L)$. With the same argument we get $K - \mathrm{acc}\, L \subseteq \mathrm{acc}(K-L)$. Therefore

$$(\mathrm{acc}\, K - L) \cup (K - \mathrm{acc}\, L) \subseteq \mathrm{acc}(K-L)\,.$$

Conversely, let $\lambda \in \mathrm{acc}(K-L)$. Then there exist a sequence $\lambda_n \in (K-L)$ such that λ_n converge to λ. Hence for each integer n, there exist $\mu_n \in K$ and $\nu_n \in L$ such that $\lambda_n = \mu_n - \nu_n$. Then from the sequence (μ_n) there exists a subsequence (μ_{n_i}) which converge to some μ. Also from the sequence (ν_n) there exists a subsequence (ν_{n_i}) which converge to some ν. Hence $\mu \in \mathrm{acc}\, K$, $\nu \in \mathrm{acc}\, L$ and $\lambda = \mu - \nu$. Thus

$$\mathrm{acc}(K-L) \subseteq (\mathrm{acc}\, K - L) \cup (K - \mathrm{acc}\, L)\,.$$

∎

Definition 3.2. An operator $T \in L(X)$ is said to be polaroid if

$$\mathrm{iso}\, \sigma(T) \subseteq \Pi(T)\,.$$

The next Lemma has been established in [19, Lemma 2.2] for Hilbert spaces operators. We show here that it holds also in the general case of Banach spaces.

In the following result we give the expression for the Browder spectrum of the elementary operator $\mathrm{d}_{A,B}$.

Lemma 3.3. *Let $A, B \in L(X)$, then*

$$\sigma_b(\delta_{A,B}) = \big(\sigma_b(A) - \sigma(B)\big) \cup \big(\sigma(A) - \sigma_b(B)\big)\,,$$

and

$$\sigma_b(\Delta_{A,B}) = \sigma(A)\sigma_b(B) \cup \sigma_b(A)\sigma(B) - \{1\}\,.$$

Proof. It is well-known that

$$\sigma(\delta_{A,B}) = \sigma(A) - \sigma(B)\,,$$
$$\sigma_e(\delta_{A,B}) = \big(\sigma_e(A) - \sigma(B)\big) \cup \big(\sigma(A) - \sigma_e(B)\big)\,.$$

For the first equality, we have

$$\begin{aligned}\sigma_b(\delta_{A,B}) &= \sigma_e(\delta_{A,B}) \cup \operatorname{acc}\sigma(\delta_{A,B})\\ &= \big[\big(\sigma_e(A) - \sigma(B)\big) \cup \big(\sigma(A) - \sigma_e(B)\big)\big]\\ &\qquad \cup \big[\operatorname{acc}\big(\sigma(A) - \sigma(B)\big)\big]\\ &= \big[\big(\sigma_e(A) - \sigma(B)\big) \cup \big(\sigma(A) - \sigma_e(B)\big)\big]\\ &\qquad \cup \big(\operatorname{acc}\sigma(A) - \sigma(B)\big)\\ &= \big[\big(\sigma_e(A) - \sigma(B)\big) \cup \big(\operatorname{acc}\sigma(A) - \sigma(B)\big)\big]\\ &\qquad \cup \big[\big(\sigma(A) - \sigma_e(B)\big) \cup \big(\sigma(A) - \operatorname{acc}\sigma(B)\big)\big]\\ &= \big(\sigma_b(A) - \sigma(B)\big) \cup \big(\sigma(A) - \sigma_b(B)\big)\,,\end{aligned}$$

where the third equality follows from Lemma 3.1.

The equality $\sigma_b(\Delta_{A,B}) = \sigma(A)\sigma_b(B) \cup \sigma_b(A)\sigma(B) - \{1\}$ follows at once from [5, Proposition 4.3 (iii)]. ∎

LEMMA 3.4. *Let A and $B \in L(H)$. Then*

$$\sigma_W(\delta_{A,B}) \subseteq \big(\sigma_W(A) - \sigma(B)\big) \cup \big(\sigma(A) - \sigma_W(B)\big)$$

and

$$\sigma_W(\Delta_{A,B}) \subseteq \sigma(A)\sigma_W(B) \cup \sigma_W(A)\sigma(B) - \{1\}\,.$$

Proof. Let $\lambda \notin (\sigma_W(A) - \sigma(B)) \cup (\sigma(A) - \sigma_W(B))$, then from [13, Theorem 3.1] $\lambda \in \sigma(\delta_{A,B}) \backslash \sigma_e(\delta_{A,B})$, it follows from [14, Theorem 4.2] and [15, Theorem 3.6] that

$$\lambda = \alpha_i - \beta_i \qquad (1 \le i \le n)\,,$$

where $\alpha_i \in \operatorname{iso}\sigma(A)$, for $1 \le i \le m$ and $\beta_i \in \operatorname{iso}\sigma(B)$, for $m+1 \le i \le n$. We have

$$\begin{aligned}\operatorname{ind}(\delta_{A,B} - \lambda I) = &\sum_{j=m+1}^{n} \dim H_0(B - \beta_j)\operatorname{ind}(A - \alpha_j)\\ &\qquad - \sum_{k=1}^{m} \dim H_0(A - \alpha_k)\operatorname{ind}(B - \beta_k)\,.\end{aligned}$$

Here $H_0(A-\alpha_k)$ and $H_0(B-\beta_j)$ denotes the quasi-nilpotent part

$$H_0(A-\alpha_k)=\left\{x\in H\ :\ \lim_{n\to\infty}\left\|(A-\alpha_k)^n x\right\|^{\frac{1}{n}}=0\right\}$$

of the operator $A-\alpha_k$.

$A-\alpha_i$ and $B-\beta_j$ are Fredholm operators and $\operatorname{ind}(A-\alpha_i)=\operatorname{ind}(B-\beta_j)=0$. Since $\alpha_i\in\operatorname{iso}\sigma(A)$ for $1\le i\le m$, and $\beta_i\in\operatorname{iso}\sigma(B)$ for $m+1\le i\le n$, it follows that $\dim H_0(A-\alpha_k)$ is finite, for $1\le i\le m$ and $\dim H_0(B-\beta_j)$ is finite for $m+1\le i\le n$. Thus $\operatorname{ind}(\delta_{A,B}-\lambda I)=0$, consequently $\lambda\notin\sigma_W(\delta_{A,B})$.

The inclusion $\sigma_W(\Delta_{A,B})\subseteq\sigma(A)\sigma_W(B)\cup\sigma_W(A)\sigma(B)-\{1\}$ follows from [5]. ∎

In the following results we give necessary and sufficient condition for $\mathrm{d}_{A,B}$ to satisfy Browder's theorem.

THEOREM 3.5. *If $A,B\in L(H)$ satisfy Browder's theorem, then the following conditions are equivalent*

(i) *$\delta_{A,B}$ satisfies Browder's theorem;*

(ii) $\sigma_W(\delta_{A,B})=\big(\sigma_W(A)-\sigma(B)\big)\cup\big(\sigma(A)-\sigma_W(B)\big)$.

Proof. Since A and B satisfy Browder's theorem then $\sigma_W(A)=\sigma_b(A)$ and $\sigma_W(B)=\sigma_b(B)$. Assume that $\sigma_W(\delta_{A,B})=\big(\sigma_W(A)-\sigma(B)\big)\cup\big(\sigma(A)-\sigma_W(B)\big)$. Then by Lemma 3.3 and Lemma 3.4

$$\begin{aligned}\sigma_b(\delta_{A,B}) &= \big(\sigma_b(A)-\sigma(B)\big)\cup(\sigma(A)-\sigma_b(B))\\ &= \big(\sigma_W(A)-\sigma(B)\big)\cup\big(\sigma(A)-\sigma_W(B)\big)\ =\ \sigma_W(\delta_{A,B})\,.\end{aligned}$$

If $\delta_{A,B}$ satisfies Browder's theorem, then

$$\begin{aligned}\sigma_b(\delta_{A,B})\ =\ \sigma_W(\delta_{A,B}) &\subseteq \big(\sigma_W(A)-\sigma(B)\big)\cup\big(\sigma(A)-\sigma_W(B)\big)\\ &\subseteq \big(\sigma_b(A)-\sigma(B)\big)\cup\big(\sigma(A)-\sigma_b(B)\big)\ =\ \sigma_b(\delta_{A,B})\,.\end{aligned}$$

∎

THEOREM 3.6. *If $A,B\in L(H)$ satisfy Browder's theorem, then the following conditions are equivalent*

(i) *$\Delta_{A,B}$ satisfies Browder's theorem;*

(ii) $\sigma_W(\Delta_{A,B})=\sigma(A)\sigma_W(B)\cup\sigma_W(A)\sigma(B)-\{1\}$.

Proof. We argue as in the proof of Theorem 3.5. ∎

EXAMPLE 3.7. For $a \in \mathbb{C}$ and $r > 0$ let $D(a, r)$ be the closed disc centered at a and with radius r. Let $\{e_n,\, n \geq 1\}$ be the usual basis of l^2. Let S be the unilateral weighted shift on l^2 defined by $S(e_n) = e_{n+1}$. Let A and B be the operators defined on $H = l^2 \oplus l^2$ by

$$A = (e_1 \otimes e_1) \oplus \left(\frac{1}{2}S^* - I\right),$$

$$B = (-e_1 \otimes e_1) \oplus \left(\frac{1}{2}S + I\right).$$

It is well known that

$$\sigma(S) = \sigma_W(S) = D(0,1)\,,$$
$$\sigma(e_1 \otimes e_1) = \sigma(e_1 \otimes e_1) = \{0,1\}\,,$$
$$\sigma_W(e_1 \otimes e_1) = \{0\}\,.$$

Then

$$\sigma(A) = \{0,1\} \cup D\Big(-1, \frac{1}{2}\Big)\,, \qquad \sigma(B) = \{-1,0\} \cup D\Big(1, \frac{1}{2}\Big)\,,$$

while

$$\sigma_W(A) = \{0\} \cup D\Big(-1, \frac{1}{2}\Big)\,, \qquad \sigma_W(B) = \{0\} \cup D\Big(1, \frac{1}{2}\Big)\,.$$

Since S satisfies SVEP, then A^* and B also satisfy the SVEP and hence satisfy the Browder's theorem by [22]. Since Browder's theorem is stable by duality [4], then A satisfies Browder's theorem. In particular, $\sigma_W(A) = \sigma_b(A)$ and $\sigma_W(B) = \sigma_b(B)$. Hence

$$\sigma_W(A)\sigma(B) \;=\; \sigma_b(A)\sigma(B) = \Big[D\Big(-1, \frac{1}{2}\Big) \cdot D\Big(1, \frac{1}{2}\Big)\Big] \cup \{0\} \cup D\Big(1, \frac{1}{2}\Big)\,,$$

$$\sigma(A)\sigma_W(B) \;=\; \sigma(A)\sigma_b(B) = \Big[D\Big(-1, \frac{1}{2}\Big) \cdot D\Big(1, \frac{1}{2}\Big)\Big] \cup \{0\} \cup D\Big(1, \frac{1}{2}\Big)\,.$$

Thus

$$\sigma_b(\Delta_{A,B}) = \Big[\Big(D\Big(-1, \frac{1}{2}\Big) \cdot D\Big(1, \frac{1}{2}\Big)\Big) \cup \{0\} \cup D\Big(1, \frac{1}{2}\Big)\Big] - 1\,.$$

Then $0 \in \sigma_b(\Delta_{A,B})$. But $0 \notin \sigma_W(\Delta_{A,B})$ by [18]. Hence

$$\sigma_W(\Delta_{A,B}) \neq \sigma_W(A)\sigma(B) \cup \sigma(A)\sigma_W(B) - 1 \,.$$

Therefore $\Delta_{A,B}$ does not satisfy Browder's theorem.

It is easily seen that, if $T \in L(X)$ is polaroid, then $\Pi(T) = E(T)$.

Theorem 3.8. *Assume that A and B are polaroid. If A and B satisfy generalized Browder's theorem, then the following assertions are equivalent.*

(i) *$L_A R_B$ satisfies generalized Browder's theorem.*

(ii) *$\sigma_{BW}(L_A R_B) = \sigma_{BW}(A)\sigma(B) \cup \sigma(A)\sigma_{BW}(B)$.*

Proof. Since A and B are polaroid, then it follows from [5, Lemma 4.7], that $L_A R_B$ is polaroid. Hence $E(L_A R_B) = \Pi(L_A R_B)$. Then, $L_A R_B$ satisfies generalized Browder's theorem if and only if it satisfies generalized Weyl's theorem, by [2, Corollary 2.1]. Now (i) is equivalent to (ii) by [19, Theorem 2.6]. ∎

If we combine last result with [5, Theorem 4.5] we get

Corollary 3.9. *Assume that A and B are polaroid. If A and B satisfy Browder's theorem (or generalized Browder's theorem), then the following assertions are equivalent.*

(i) *$L_A R_B$ satisfies generalized Browder's theorem;*

(ii) *$\sigma_{BW}(L_A R_B) = \sigma_{BW}(A)\sigma(B) \cup \sigma(A)\sigma_{BW}(B)$;*

(iii) *$L_A R_B$ satisfies Browder's theorem;*

(iv) *$\sigma_W(L_A R_B) = \sigma_W(A)\sigma(B) \cup \sigma(A)\sigma_W(B)$.*

Theorem 3.8 gives a partial positive answer to the first question posed in [5, Remark 4.6]. However, in the general case the answer is negative as shown by the following example:

Example 3.10. Let A be a nonzero nilpotent operator ($A^{p-1} \neq 0 = A^p$ for some integer $p > 1$). Let B be a quasinilpotent which is not nilpotent. Here A is polaroid and B is not. Also A and B satisfies Browder's and generalized Browder's theorems. It is not difficult to see that

$$\sigma(A) = \{0\}\,, \qquad \sigma_{BW}(A) = \emptyset \qquad \text{and} \qquad \sigma(B) = \sigma_{BW}(B) = \{0\}\,.$$

Hence

$$\sigma_{BW}(A)\sigma(B) \cup \sigma(A)\sigma_{BW}(B) = \{0\}\,.$$

But since $L_A R_B$ is nilpotent $((L_A R_B)^p = 0)$ then 0 is a pole and then $\sigma_{BW}(L_A R_B) = \emptyset$. Here $L_A R_B$ satisfies Browder's and generalized Browder's theorems.

Remark. Let A and B satisfy Browder's theorem. It follows from Example 3.10 that in general, equality $\sigma_W(L_A R_B) = \sigma_W(A)\sigma(B) \cup \sigma(A)\sigma_W(B)$ does not imply $\sigma_{BW}(L_A R_B) = \sigma_{BW}(A)\sigma(B) \cup \sigma(A)\sigma_{BW}(B)$.

4. Application

A Banach space operator $T \in L(X)$ is said to be hereditarily normaloid, $T \in \mathcal{HN}$, if every part of T (i.e., the restriction of T to each of its invariant subspaces) is normaloid (i.e., $\|T\|$ equals the spectral radius $r(T)$), $T \in \mathcal{HN}$ is totally hereditarily normaloid $\mathcal{THN}$ if also the inverse of every invertible part of T is normaloid and T is completely (totally) hereditarily normaloid $T \in \mathcal{CHN}$, if either $T \in \mathcal{THN}$ or $T - \lambda I \in \mathcal{HN}$ for every complex number λ. The class $\mathcal{CHN}$ is large. In particular, Hilbert space operators T, which are either hyponormal $\big(T^*T \geq TT^*\big)$ or p-hyponormal $\big((T^*T)^p \geq (TT^*)^p$ for some $0 < p \leq 1\big)$ or w-hyponormal $\Big(\big(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}}\big)^{\frac{1}{2}} \geq |T^*|\Big)$ are $\mathcal{THN}$ operators. Again totaly *-paranormal operators $\big(\|(T-\lambda I)^*x\|^2 \leq \|(T-\lambda I)x\|^2$ for every unit vector $x\big)$ are $\mathcal{HN}$ operator and paranormal operators $\big(\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all unit vector $x\big)$ are $\mathcal{THN}$ operators.

It is proved in [12] that if $A, B^* \in L(H)$ are log-hyponormal or p-hyponormal operators, then generalized Weyl's theorem holds for $f(\mathrm{d}_{A,B})$ for every $f \in \mathcal{H}(\sigma(\mathrm{d}_{A,B}))$, where $\mathcal{H}(\sigma(\mathrm{d}_{A,B}))$ is the set of all analytic functions defined on a neighborhood of $\sigma(\mathrm{d}_{A,B})$. This result was extended by [8] and [21] to w-hyponormal operators. In [9, Theorem 4.3] it proved that if $A, B \in \mathcal{THN}$, with the additional conditions $\ker B \subseteq \ker B^*$ and $\mathrm{d}_{A,B}$ has SVEP, then Weyl's theorem holds for $f(\mathrm{d}_{A,B})$ and $f(\mathrm{d}^*_{A,B})$. In the next results we can give more.

THEOREM 4.1. *Suppose that $A, B \in L(H)$ are $\mathcal{CHN}$ operators, then*

$$\sigma_W(\delta_{A,B}) = \big(\sigma_W(A) - \sigma(B)\big) \cup \big(\sigma(A) - \sigma_W(B)\big)\,,$$

and

$$\sigma_W(\Delta_{A,B}) = \sigma(A)\sigma_W(B) \cup \sigma_W(A)\sigma(B) - \{1\}\,.$$

Proof. Let $\lambda \in \sigma(\delta_{A,B}) \setminus \sigma_W(\delta_{A,B})$. Then $\lambda = \mu - \nu$, such that $\mu \in \sigma(A)$, $\nu \in \sigma(B)$ and $\delta_{A,B} - \lambda I$ is a Fredholm operator of index 0, from [13, Theorem 3.1] we have $A - \mu I$ and $B - \nu I$ are Fredholm operators, and from [10, Theorem 2.9] we have A and A^* have SVEP on the complement of $\sigma_W(A)$, it follows from [1, Corollary 3.21] that $\mu \in \Pi(A)$. Similarly we get $\nu \in \Pi(B)$.

Since Browder's theorem holds for $\mathcal{CHN}$ operators [10, Corollary 2.15], consequently $\mu \notin \sigma_b(A) = \sigma_W(A)$, $\nu \notin \sigma_b(B) = \sigma_W(B)$ and

$$\big(\sigma_W(A) - \sigma(B)\big) \cup \big(\sigma(A) - \sigma_W(B)\big) \subseteq \sigma_W(\delta_{A,B})\,.$$

The other inclusion holds from Lemma 3.4, then

$$\sigma_W(\delta_{A,B}) = \big(\sigma_W(A) - \sigma(B)\big) \cup \big(\sigma(A) - \sigma_W(B)\big)\,.$$

Similarly we get $\sigma_W(\Delta_{A,B}) = \sigma(A)\sigma_W(B) \cup \sigma_W(A)\sigma(B) - \{1\}$. ∎

Corollary 4.2. *Suppose that $A, B \in L(H)$ are $\mathcal{CHN}$ operators, then $\mathrm{d}_{A,B}$ has SVEP at points $\lambda \notin \sigma_W(\mathrm{d}_{A,B})$.*

Proof. By Theorem 3.5 and Theorem 4.1, we get Browder's theorem holds for $\mathrm{d}_{A,B}$ and from [3, Proposition 2.2] $\mathrm{d}_{A,B}$ has SVEP at points $\lambda \notin \sigma_W(\mathrm{d}_{A,B})$. ∎

Corollary 4.3. *Suppose that $A, B \in L(H)$ are $\mathcal{CHN}$ operators, then generalized Weyl's theorem holds for $f(\mathrm{d}_{A,B})$ and $f(\mathrm{d}^*_{A,B})$ for every $f \in \mathcal{H}(\sigma(\mathrm{d}_{A,B}))$, where $\mathrm{d}^*_{A,B}$ is the dual of $\mathrm{d}_{A,B}$.*

Proof. By Theorem 3.5 and Theorem 4.1, Browder's theorem holds for $\mathrm{d}_{A,B}$ this is equivalent to Browder's theorem holds for $\mathrm{d}^*_{A,B}$. Recall from [10, Proposition 2.1] that A and B are polaroid, it follows from [19, Lemma 2.2] that $\mathrm{d}_{A,B}$ is polaroid, hence $E(\mathrm{d}_{A,B}) = \Pi(\mathrm{d}_{A,B})$. Then it follows by [2, Corollary 2.1] that generalized Weyl's theorem holds for $\mathrm{d}_{A,B}$. Since $\mathrm{d}_{A,B}$ is polaroid, then it is isoloid, i.e., every isolated point of the spectrum is an eigenvalue of $\mathrm{d}_{A,B}$. From [24, Theorem 2.2] it follows that generalized Weyl's theorem holds for $f(\mathrm{d}_{A,B})$.

Since $\mathrm{d}_{A,B}$ is polaroid, then $\mathrm{d}^*_{A,B}$ is also polaroid, hence $E(\mathrm{d}^*_{A,B}) = \Pi(\mathrm{d}^*_{A,B})$, and Browder's theorem holds for $\mathrm{d}^*_{A,B}$, we argue as above we get $f(\mathrm{d}^*_{A,B})$ satisfies generalized Weyl's theorem for every $f \in \mathcal{H}(\sigma(\mathrm{d}_{A,B}))$. ∎

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